MATH 64091
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Pick's Formula

Let's call a point in he plane a *lattice* (or an *integer*) point if both of its coordinates are integers. Let P be a *lattice* polygon in the plane, that is, a polygon all of whose vertices are lattice points. Let I be the number of lattice points that are strictly inside P and B be the number of lattice points that are on the boundary of P. The purpose of this project is to prove Pick's formula which expresses the area A of P in terms of I and B:

$$A = I + \frac{B}{2} - 1$$

Problem 1. Pick a complicated enough lattice polygon P. Compute I, B, and A (directly, not using Pick's formula). Check that Pick's formula is satisfied.

Problem 2. Let P be a lattice rectangle with sides parallel to the coordinate axes. We can then assume that the vertices of P are at the points $(0,0)$, $(a,0)$, $(0,b)$, and (a,b) for some integers a and b. Check that Pick's formula holds for P.

Problem 3. Now let T be a lattice triangle with two sides parallel to the coordinate axes. We can then assume that the vertices of T are at the points $(0,0)$, $(a,0)$, and $(0,b)$ for some integers a and b. Check that Pick's formula holds for T. Hint: Consider a rectangle P from previous problem. Let I_P, B_P and A_P be the numbers of interior lattice points, boundary lattice points, and the area for P, while I_T, B_T, and A_T be the corresponding parameters for T. Let c be the number of lattice points on the hypothenuse of T (not counting the vertices). From Problem 2, we already know that $A_P = I_P + B_P/2 - 1$. Express A_P, I_P, and B_P in terms of I_T, B_T, A_T, and c. Plug into Pick's formula for P. Obtain Pick's formula for T.

Problem 4. Next, let T be an arbitrary lattice triangle. Check that Pick's formula holds for T. Hint: Consider a rectangle P whose sides are parallel to the coordinate axes, such that P shares one of the vertices with T and two other vertices of T are on the sides of P. Draw the picture. Notice that P is broken into four triangles, one of them is T, and there are three more, all of the kind considered in Problem 3. We already know that Pick's formula holds for P and the three triangles. Expressing the parameters for P in terms of the parameters for the triangles, prove Pick's formula for T.

Problem 5. Assume Pick's formula holds for a polygon P. Show that it holds for the polygon $P \cup T$, where T is a triangle and P and T share a side. Conclude that Pick's formula holds for any lattice polygon (not necessarily convex). Hint: This is very similar to Problem 2. Denote the number of lattice points of the common side by c.

Problem 6. Let M be the centroid of a triangle ABC . Show that if you connect M to the vertices of ABC you will break ABC into three triangles of equal area. Show that no other point N inside ABC has this property.

Problem 7. Let ABC be a lattice triangle that has just one integer point inside and the only lattice points on its boundary are the vertices. Show that this interior lattice point is the centroid of ABC. Hint: Use Picks formula ($A = I + B/2 - 1$) and the previous problem.

Problem 8. Pick's formula, in particular, says, that in order for a polygon to have a large area, it needs to have a large overall number of lattice points inside and on the boundary. Show that this is not the case in 3D, that is, a 3D polytope with fixed I and B can have huge volume. For this, consider the tetrahedron with the vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, c)$, where c is a positive integer.

Problem 9. Let P be a lattice polygon with a few lattice holes. Can you adjust Pick's formula so that it works for such a polygon?